| Student No: | |
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| | |

# Mathematics Extension 2

## Year 12, Assessment task 3, Term 2 2024

**General Instruction:**

- Reading time – 10 minutes
- Working time – 180 minutes
- Write using black/blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

**Total marks:**

**Section I – 10 marks (pages 1–3)**

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II – 90 marks (pages 4–9)**

- Attempt Questions 11–16
- Allow about 2 hour and 35 minutes for this section

**Class Teacher (please tick one name)**

- ◯ Mr Berry
- ◯ Ms Lee
- ◯ Mr Umakanthan
- ◯ Dr Vranešević

| Question No. | 1-10 | 11 | 12 | 13 | 14 | 15 | 16 | Total |
|---|---|---|---|---|---|---|---|---|
| Marks | /10 | /15 | /15 | /15 | /15 | /15 | /15 | /100 |

## Section I

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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1. Which of the following is **NOT** equivalent to ‘If $n$ is even, then $n^2$ is a multiple of 4’?

   A) $n$ is even $\therefore n^2$ is a multiple of 4;

   B) $n^2$ is a multiple of 4 if $n$ is even;

   C) $n^2$ is a multiple of 4 $\therefore n$ is even;

   D) $n$ is even implies that $n^2$ is a multiple of 4.

2. Let $I_n = \displaystyle\int x^n e^{ax}\,dx$. Which of the following is the correct expression for $I_n$?

   A) $I_n = \dfrac{x^n e^{ax}}{a} - nI_{n-1}$;

   B) $I_n = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}I_{n-1}$;

   C) $I_n = \dfrac{x^n e^{ax}}{a} + nI_{n-1}$;

   D) $I_n = \dfrac{x^n e^{ax}}{a} + \dfrac{n}{a}I_{n-1}$.

3. The acceleration of a particle moving in a straight line with velocity $v$ is given by $\ddot{x} = v^2$. Initially $v = 1$. What is $v$ as a function of $t$?

   A) $v = 1 - t$;

   B) $v = \ln|1-t|$;

   C) $v = \dfrac{t}{1-t}$;

   D) $v = \dfrac{1}{1-t}$.

4. The negation of the statement $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $2x + 3y = 12$ is:

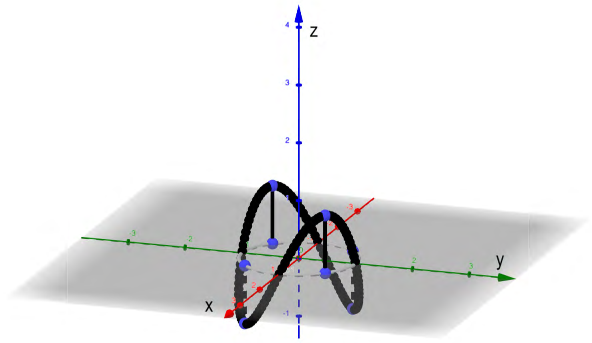
   A) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $2x + 3y \neq 12$;

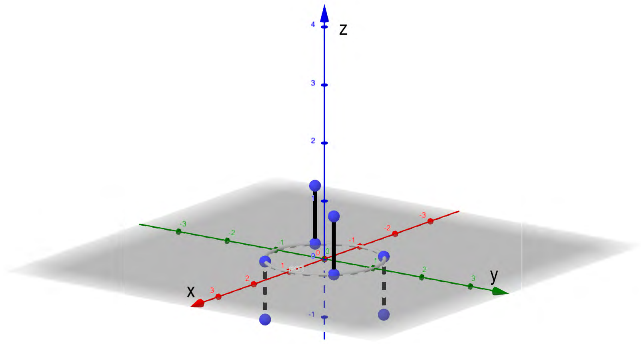
   B) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$ $2x + 3y \neq 12$;

   C) $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$ $2x + 3y \neq 12$;

   D) $\exists x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $2x + 3y \neq 12$.

5. The sketch of the curve in the shape of saddle is drawn on the left. The curve has two peaks at $t = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$ and two troughs at $t = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$, shown on the right. Which of following parametric equations are represented by the given curve:




A) $x = \sin t,\, y = \cos 2t,\, z = \sin t$;

B) $x = \sin t,\, y = \cos t,\, z = \sin 2t,$;

C) $x = \sin 2t,\, y = \cos t,\, z = \sin t$;

D) $x = \sin 2t,\, y = \cos t,\, z = \sin 2t$.

6. Which of the following complex numbers equals $(\sqrt{3} + i)^4$

A) $-2 + \dfrac{2}{\sqrt{3}} i$;

B) $-8 + \dfrac{8}{\sqrt{3}} i$;

C) $-2 + 2\sqrt{3} i$;

D) $-8 + 8\sqrt{3} i$.

7. The line $l_1$ has vector equation $\mathbf{r_1} = \mathbf{i} + \lambda(\mathbf{j} - \mathbf{k})$ and the line $l_2$ has vector equation $\mathbf{r}_2 = (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \mu(2\mathbf{i} + \mathbf{k})$. Which of the following statements is correct?

A) $l_1$ and $l_2$ are parallel;

B) $l_1$ and $l_2$ are perpendicular;

C) $l_1$ and $l_2$ intersect at a point;

D) $l_1$ and $l_2$ are skew.

8. Which of the following is an expression for $\displaystyle\int \frac{4x^2 - 5x - 1}{(x-3)(x^2+1)}\, dx$

A) $\ln[(x-3)(x^2+1)] + C$;

B) $\ln[(x-3)^2(x^2+1)] + C$;

C) $\ln[(x-3)(x^2+1)] + \tan^{-1} x + C$;

D) $\ln[(x-3)^2(x^2+1)] + \tan^{-1} x + C$.

9. What is the solution to the equation $z^2 = i\bar{z}$

   A) $(0,0)$ and $(0,1)$;

   B) $(0,0)$ and $(0,-1)$;

   C) $(0,0)$, $(0,-1)$, $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $(\frac{\sqrt{3}}{2}, \frac{1}{2})$;

   D) $(0,0)$, $(0,1)$, $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $(\frac{\sqrt{3}}{2}, \frac{1}{2})$;

10. A particle of unit mass $m$ is projected vertically upwards with an initial velocity of $u$ ms$^{-1}$ in a medium in which the resistance to the motion is proportional to the square of the velocity $v$ ms$^{-1}$ of the particle or $mkv^2$. Let $x$ be the displacement in metres of the particle above the point of projection, O, so that the equation of motion is $\ddot{x} = -(g + kv^2)$ where $g$ ms$^{-2}$ is the acceleration due to gravity and $k$ is constant proportional to $g$, given as $k = 0.01g$.
    Which of the following gives the correct expression for the time taken?

    A) $t = \frac{100}{g}\left(\tan^{-1}\frac{u}{10} - \tan^{-1}\frac{v}{10}\right)$;

    B) $t = \frac{10}{g}\left(\tan^{-1}u - \tan^{-1}v\right)$;

    C) $t = \frac{10}{g}\left(\tan^{-1}\frac{u}{10} - \tan^{-1}\frac{v}{10}\right)$;

    D) $t = \frac{100}{g}\left(\tan^{-1}u - \tan^{-1}v\right)$.

**End of Section I**

**Section II**

**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hour and 35 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11**. (15 marks) Use the Question 11 Writing Booklet

(a) State whether each statement is true or false, justifying your answer.

(i) $\forall n \in \mathbb{N}, n^2 > n$ **1**

(ii) $\exists x \in \mathbb{R}$ such as $x^2 + 1 = 0$ **1**

(iii) $\forall x \in \mathbb{R}, \exists y \in \mathbb{N}$ such as $x + y = 10$ **1**

(b) It is given that $f(z) = z^4 + 2\sqrt{2}z^3 + z^2 + 8\sqrt{2}z - 12$ . One of the roots of the equation $f(z) = 0$ is given by $2i$. By factorising $f(z)$ as a product of two quadratic factors, obtain the other roots of the equation. **3**

(c) Find the integral:

$$\int \frac{e^{\arctan x} + x\ln(1 + x^2) + 1}{1 + x^2} dx$$ **2**

(d) The displacement of a particle moving in a straight line, $s$ m, at time $x$ seconds is given by $s = \sin 4x + 2\sin 2x + 2$.

(i) Show that the velocity of the particle at time $x$ seconds is given by $v = \dfrac{8}{(1+t^2)^2}(1 - 3t^2)$ ms$^{-1}$, where $t = \tan x$ **2**

(ii) Hence find the value of $x$ where $0 \leq x \leq \pi$ for which the displacement is maximised. **2**

(e) Find the solutions of the integral equation with respect to $a$, for $2 \leq a \leq 3$, of

$$\int_0^a \cos(x + a^2)dx = \sin a$$ **3**

**Question 12.** (15 marks) Use the Question 12 Writing Booklet

(a) A complex number $z_1$ is given by $z_1 = a + (a-3)i$, where $a$ is a positive real constant. It is given that $\arg z_1 = \theta$, where $-\dfrac{\pi}{2} < \theta < 0$.

(i) Find, leaving your answer in terms of $\theta$,

($\alpha$) $\arg(-2z_1)$, **1**

($\beta$) $\arg(z_1 - 2a)$. **2**

(ii) Another complex number $z_2$ is given by $z_2 = 1 + 3i$. Without using a calculator, find the range of values of $a$ such that

$$\frac{|z_1 z_2|^2}{\mathrm{Im}(z_1 z_2)} \leq 10$$

**4**

(b) Prove that a four-digit number is divisible by 11 if and only if the difference between the sum of its even digits and the sum of its odd digits is a multiple of 11 (including zero).
**Hint**: Divide 1001 by 11. **3**

(c) (i) Differentiate $\cot^3 x$ **1**

(ii) Differentiate $\cot^5 x$ **1**

(iii) Using derivatives from (i) and (ii) find the integral of

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^6 x \, dx$$

**3**

**Question 13.** (15 marks) Use the Question 13 Writing Booklet

(a) (i) if $k$ is an integer where $k \geq 3$ and $(k-1)(k+1) < k^2$, show that

$$\frac{1}{(k-1)k(k+1)} > \frac{1}{k^3}$$ **1**

(ii) Given that $S_n = \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots + \frac{1}{n^3} = \sum_3^n \frac{1}{k^3}$, use partial fractions from previous part or otherwise to prove that $S_n < \frac{1}{12}$. **3**

(b) Let $p$ and $q$ be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that

(i)

$$\frac{1}{3} \leq \frac{1}{p(p+1)} + \frac{1}{q(q+1)} \leq \frac{1}{2}$$ **2**

(ii)

$$\frac{1}{p(p-1)} + \frac{1}{q(q-1)} \geq 1$$ **2**

(c) An object is moving on a horizontal plane and at position $x$ metres from the origin it has acceleration ($\ddot{x}$ m s$^{-2}$) given by $\ddot{x} = 0.01\left(x + \frac{64}{x^3}\right)$. Initially the object is 4 metres to the left of the origin and moving to the left at a speed of $\frac{\sqrt{3}}{5}$ m s$^{-1}$.

(i) If the velocity of the object is $v$ m s$^{-1}$, show that

$$v^2 = 0.01\left(\frac{x^4 - 64}{x^2}\right).$$ **2**

(ii) Explain why the velocity of the object at a position $x$ metres from the origin is given by

$$v = \frac{\sqrt{x^4 - 64}}{10x}.$$ **1**

(iii) Find, correct to the nearest second, the time to reach a position 50 metres to the left of the origin. **4**

**Question 14.** (15 marks) Use the Question 14 Writing Booklet

(a) Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$(a_1^2 + a_1)(a_2^2 + a_2)\ldots(a_n^2 + a_n) \geq 2^n.$$

**3**

(b) (i) A parachutist of mass 90 kg jumps out of an aeroplane at a height of 100 m. The acceleration due to gravity is 10 m/s$^2$.
Derive the vertical velocity functions $v$ m/s$^2$ in terms of displacement $x$ for the parachutist, if the initial velocity is $u$ m/s$^2$ and there is:

($\alpha$) **no** resistance force during free fall; **2**

($\beta$) a resistance force of $0.27v^2$ Newtons when parachute is open. **3**

(ii) A parachutist jumps from rest, and opens his parachute 60 m from the ground.

($\alpha$) Find the vertical velocity when parachute is open. **1**

($\beta$) Find the vertical velocity (1 decimal place) on landing. **2**

($\gamma$) What percentage (to the nearest whole percent) is the landing velocity compared to the terminal velocity? **2**

(c) Let $a, b, c$ be positive integers. Prove that it is impossible for all three numbers $a^2 + b + c$, $b^2 + a + c$, $c^2 + a + b$ to be perfect squares.

**2**

**Question 15.** (15 marks) Use the Question 15 Writing Booklet

(a) Consider the following vector equation $\mathbf{r} = (2+\lambda)\mathbf{i} + (3-\lambda)\mathbf{j} + (4-2\lambda)\mathbf{k}$.

(i) Find the coordinates of the points where the line $\mathbf{r}$ cuts the coordinate planes, where point $A$ is on $xy$ plane, point $B$ is on $xz$ plane, and point $C$ is on $yz$ plane. **3**

(ii) Find the vector projection of $\overrightarrow{OC}$ onto $\overrightarrow{AB}$ **1**

(iii) Find the area of $\Delta OAC$ correct to 3 significant figures. **2**

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \left(\sin x\right)^n \, dx$, where $n$ is an integer $n \geq 0$.

(i) Using integration by parts, show that for $n \geq 2$, **3**

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

(ii) Deduce that **3**

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \ldots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{and} \quad I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \ldots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

(iii) Explain why **1**

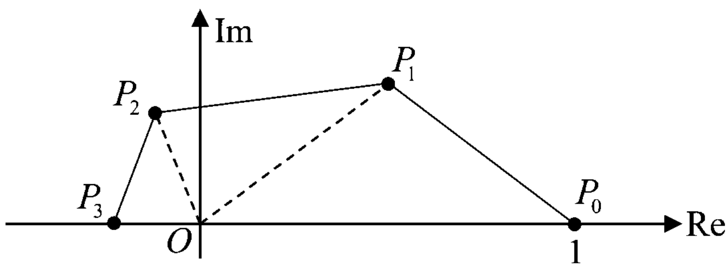
$$I_k > I_{k+1}$$

(iv) Hence, using the fact that $I_{2n-1} > I_{2n}$ and $I_{2n} > I_{2n+1}$, show that **2**

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2 \cdot 4^2 \cdot \ldots (2n)^2}{1 \cdot 3^2 \cdot 5^2 \cdot \ldots (2n-1)^2 \cdot (2n+1)} < \frac{\pi}{2}$$

**Question 16.** (15 marks) Use the Question 16 Writing Booklet

(a) A complex number $z$ is given by $z = re^{i\frac{\pi}{n}}$, where $0 < r < 1$ and $n$ is a positive integer with $n \geq 3$. The numbers $1,\ z,\ z^2,\ \dots,\ z^n$ can be represented by the points $P_0,\ P_1,\ P_2,\ \dots,\ P_n$ respectively in an Argand diagram. The $(n+1)$-sided polygon formed by using $P_0,\ P_1,\ P_2,\ \dots,\ P_n$ is called the $(n+1)$-*polygon generated by* $z$. An example of a $(3+1)$-polygon generated by $z$ is shown in the following Argand diagram (not drawn to scale).




Let $z = \dfrac{1}{4}(1+\sqrt{3}i)$ for parts (i) to (iii).

(i) Express $z$ in the form $re^{i\theta}$ where $r > 0$ and $0 < \theta \leq \pi$. **2**

(ii) Hence write down $z^2$ and $z^3$ in similar form. **2**

(iii) Find the exact area of triangle $OP_0P_1$, where $O$ is the origin. Hence find the exact area of the $(3+1)$-polygon generated by $z$. **4**

Let $z = \dfrac{1}{2}re^{i\frac{\pi}{n}}$ for parts (iv)

(iv) Find the area of an $(n+1)$- polygon generated by $z$ in terms of $n$, leaving your answer in the form $a(1-b^n)\sin\dfrac{\pi}{n}$, where $a$ and $b$ are real numbers to be determined. **3**

(b) (i) Consider triangle $ABC$, where $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$. Find a vector (in terms of $\mathbf{b}$ and $\mathbf{c}$) which bisects the angle $\angle CAB$. **1**

(ii) On the same triangle $ABC$, $D$ and $E$ are points on the sides $AC$ and $AB$ of the triangle, respectively. Also, $DE$ is not parallel to $CB$. Suppose $F$ and $G$ are points of $BC$ and $ED$, respectively, such that $BF : FC = EG : GD = BE : CD$. Show that $GF$ is parallel to the angle bisector of $\angle CAB$. **3**

**End of Examination**

Q1 A) n is even $\Rightarrow$ $n^2$ is multiple of 4
B) $n^2$ is a multiple of 4 if n is even
C) $n^2$ is a multiple of 4 $\Rightarrow$ n is even
D) n is even implies $n^2$ is a multiple of 4
$\therefore$ C

Q2 $I_n = \int x^n e^{ax}\,dx$ (by parts)

$u = x^n \qquad v' = e^{ax}$

$u' = n x^{n-1} \qquad v = \frac{1}{a} e^{ax}$

$= \frac{x^n e^{ax}}{a} - \int n x^{n-1} \times \frac{1}{a} e^{ax}\,dx$

$= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} a^{ax}\,dx$

$\therefore I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$

B

Q3 $\ddot{x} = v^2$, $t=0, v=1$

$v = f(t) = ?$ (Law. equation of motion)

$\ddot{x} = \frac{dv}{dt} = v^2$

$dt = \frac{dv}{v^2}$ Q3 continues

$$\int_0^t dt = \int_1^v v^{-2}\,dv$$

$$t = (-1)\, v^{-1} \Big|_1^v$$

$$= (-1)\left(\frac{1}{v} - \frac{1}{1}\right)$$

$$= (-1)\,\frac{1-v}{v}$$

$$t = \frac{v-1}{v}$$

$$tv = v-1$$

$$v(t-1) = -1$$

$$v = \frac{1}{1-t}$$

D

Q4. $\neg(\forall x \in R, \exists y \in \mathbb{R}$ such that $2x+3y=12)$

$\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$ $2x+3y \neq 12$

$\therefore$ B

(quantifiers)

Q6. $(\sqrt{3}+i)^4 = \left[2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)\right]^4$

$= 2^4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^4$

$= 2^4\left(\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6}\right)$ (CN-powers De-M)

$= 16\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$= 16\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$= -8+8\sqrt{3}i$

D

Q7 $\underset{\sim}{r_1} = \hat{\underset{\sim}{i}} + \lambda(\hat{\underset{\sim}{j}} - \hat{\underset{\sim}{k}}) = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\-1\end{pmatrix}$

$\underset{\sim}{r_2} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\0\\1\end{pmatrix}$ (vector equation of straight line)

$\begin{pmatrix}1\\\lambda\\-\lambda\end{pmatrix} = \begin{pmatrix}3+2\mu\\2\\-1+\mu\end{pmatrix} \Rightarrow \lambda = 2$

$-2 = -1+\mu \Rightarrow \mu = -1$

$1 = 3+2(-1) = 1$

$\therefore$ $\underset{\sim}{r_1}$ & $\underset{\sim}{r_2}$ has a point of intersection

C

Q5 for $t = \frac{\pi}{4}$

A) $x = \frac{\sqrt{2}}{2}$, $y = 1$, $z = \frac{\sqrt{2}}{2}$ X

B) $x = \frac{\sqrt{2}}{2}$, $y = \frac{\sqrt{2}}{2}$, $z = 1$

$t = \frac{5\pi}{4}$ : $x = -\frac{\sqrt{2}}{2}$, $y = -\frac{\sqrt{2}}{2}$, $z = 1$ } for peaks

$t = \frac{3\pi}{4}$ : $x = -\frac{\sqrt{2}}{2}$, $y = \frac{\sqrt{2}}{2}$, $-1$

$t = \frac{7\pi}{4}$ : $x = \frac{\sqrt{2}}{2}$, $y = -\frac{\sqrt{2}}{2}$, $-1$ } for throughts

$\therefore$ B

Q8 $\int \frac{4x^2-5x-1}{(x-3)(x^2+1)}\,dx$ (partial fraction)

$$\frac{4x^2-5x-1}{(x-3)(x^2+1)} = \frac{a}{x-3} + \frac{bx+1}{x^2+1}$$

$$4x^2-5x-1 = a(x^2+1) + (bx+1)(x-3)$$
$$= (a+b)x^2 + (1-3b)x + a-3$$

$\boxed{a+b=4} \Rightarrow$

$-5 = 1-3b \Rightarrow -6 = -3b \Rightarrow \boxed{b=2}$

$-1 = a-3 \Rightarrow a = 2$

$$\therefore \int \left( \frac{2}{x-3} + \frac{2x+1}{x^2+1} \right) dx =$$

$$= \int \frac{2}{x-3}\, dx + \int \frac{2x\,dx}{x^2+1} + \int \frac{dx}{1+x^2}$$

$$= 2\ln|x-3| + \ln|x^2+1| + \tan^{-1} x + C$$

$$= \ln\left[(x^2+1)(x-3)^2\right] + \tan^{-1} x + C$$

$\therefore$ D

Q9. $z^2 = i\bar{z}$ $\quad z = x+iy \quad \bar{z} = x - iy$

(CN arithmetics equations)

$$z^2 = i\bar{z}$$

$$(x+iy)^2 = i(x-iy)$$

$$x^2 - y^2 + 2xyi = +y + xi$$

$$x^2 - y^2 = y \quad (1)$$

$$2xy = x \quad (2) \Rightarrow x(2y-1) = 0 \Rightarrow x=0 \text{ or } y = \frac{1}{2}$$

subs into (1)

for $x=0$ : $-y^2=y$

$y(1+y)=0 \Rightarrow y=0$ or $y=-1$

for $y=\frac{1}{2}$ : $x^2-\frac{1}{4}=\frac{1}{2}$

$x^2=\frac{3}{4} \Rightarrow x=\pm\frac{\sqrt{3}}{2}$

$\therefore (0,0),(0,-1),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

$\therefore$ C

Q10 $m=1$, $v_0=4$, $R=mkv^2$, $g$, $k=0.01g$

$m\ddot{x}=-mg-mkv^2$

$\ddot{x}=-(g+0.01gv^2)$

$\frac{dv}{dt}=-g(1+0.01v^2)$

$\frac{dt}{dv}=\frac{-1}{g}\frac{1}{1+0.01v^2}$, $(0.1)^2=0.01$

$\int_0^t dt=-\frac{1}{g}\int\frac{dv}{1+(0.1v)^2}$

$$t = \frac{-1}{g}\frac{1}{0.1}\int_u^v \frac{0.1\,dv}{1+(0.1v)^2}$$

$$= \frac{-10}{g}\left[\tan^{-1} 0.1v\right]_u^v = \frac{-10}{g}\left[\tan^{-1}\frac{v}{10}\right]_u^v$$

$$= \frac{-10}{g}\left[\tan^{-1}\frac{v}{10} - \tan^{-1}\frac{u}{10}\right]$$

$$= \frac{10}{g}\left(\tan^{-1}\frac{u}{10} - \tan^{-1}\frac{v}{10}\right)$$

$\therefore$ (C)

| | |
|---|---|
| Q1 | C |
| Q2 | B |
| Q3 | D |
| Q4 | B |
| Q5 | B |
| Q6 | D |
| Q7 | C |
| Q8 | D |
| Q9 | C |
| Q10 | C |

Q 11 a)

i) $\forall n \in N, n^3 > n$ False as for $n=1$, $1>1$ not true ①

ii) $\exists x \in R$, such as $x^2+1=0$ False as ①

$x^2=-1 \Rightarrow x=\sqrt{-1}=\pm i \notin R$

iii) $\forall x \in R, \exists y \in N$ such as $x+y=10$ False ①

as for $x=\frac{1}{2}$, $y=9\frac{1}{2} \notin N$

b) $f(z)=z^4+2\sqrt{2}z^3+z^2+8\sqrt{2}z-12$

$f(z)=0$ for $z=2i$ find other roots

since all the coef. of $f(z)$ are real: then

$f(2i)=0$, then $f(-2i)=0$ ∴ $-2i$ is another root ①

Thus product of quadratic factors =

$[z-(2i)][z-(-2i)]$

$= z^2+4$

$\therefore f(z)=(z^2+4)(z^2+Az-3)$

$= z^4+4z^2+Az^3+4Az-3z^2-12$ ①

$= z^4+Az^3+z^2+4Az-12$ $\therefore$ $A=2\sqrt{2}$ or $4A=8\sqrt{2}$ $\Rightarrow A=2\sqrt{2}$

$\Rightarrow A = 2\sqrt{2}$

$\therefore f(z) = (z^2+4)(z^2+2\sqrt{2}z-3)$

$$z^2+2\sqrt{2}z-3=0$$

$$z_{1,2} = \frac{-2\sqrt{2} \pm \sqrt{8-4\times1\times(-3)}}{2}$$

$$= \frac{-2\sqrt{2} \pm \sqrt{8+12}}{2}$$

$$= \frac{-2\sqrt{2} \pm 2\sqrt{5}}{2}$$

$$= -\sqrt{2} \pm \sqrt{5}$$

1

$\therefore$ roots are $2i, -2i, -\sqrt{2}+\sqrt{5}, -\sqrt{2}-\sqrt{5}$

e)

$$\int \frac{e^{\arctan x} + x\ln(1+x^2) + 1}{1+x^2}\,dx =$$

$$= \int \frac{e^{\arctan x}}{1+x^2}\,dx + \int \frac{x\ln(1+x^2)}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

Sub. $\arctan x = t$ & $\ln(1+x^2) = u$

$$\frac{dx}{1+x^2} = dt \qquad \frac{2x\,dx}{1+x^2} = du$$

1

$$= \int e^t\,dt + \frac{1}{2}\int u\,du + \arctan x + C$$

$$= e^t + \frac{1}{2}\,\frac{u^2}{2} + \arctan x + C$$

1

$$= e^{\arctan x} + \frac{1}{4}\left[\ln(1+x^2)\right]^2 + \arctan x + C$$

d) $s(x) = \sin 4x + 2\sin 2x + 2$

$x$ - time [sec]

i) $v = \frac{8}{(1+t^2)^2}(1-3t^2)\ \frac{m}{s}$ , $t = \tan x$

$$\frac{ds}{dx} = 4\cos 4x + 4\cos 2x$$

$t = \tan x$

$$\left.\begin{aligned} &\cos 2x = \frac{1-t^2}{1+t^2} \qquad \sin 2x = \frac{2t}{1+t^2} \\ &\cos 2\times 2x = \cos^2 2x - \sin^2 2x \end{aligned}\right\} \quad 1$$

$$\therefore \frac{ds}{dx} = v = 4\times\left(\left(\frac{1-t^2}{1+t^2}\right)^2 - \left(\frac{2t}{1+t^2}\right)^2\right)$$

$$+ 4\times\frac{1-t^2}{1+t^2}$$

$$= 4\ \frac{1-2t^2+t^4-4t^2}{(1+t^2)^2} + \frac{4-4t^2}{1+t^2}$$

$$= \frac{4-24t^2+\cancel{4t^4}+4+\cancel{4t^2}-\cancel{4t^2}-\cancel{4t^4}}{(1+t^2)^2}$$

$$= \frac{8 - 24t^2}{(1+t^2)^2} = \frac{8(1-3t^2)}{(1+t^2)^2}$$

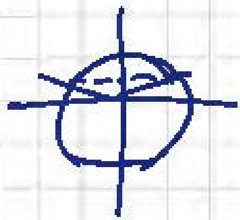
1

ii) $0 \le x \le \pi$ $\quad s_{max} = ?$

for $s_{max}$, $v = 0$

$\therefore 1 - 3t^2 = 0 \Rightarrow t^2 = \frac{1}{3} \Rightarrow t = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$

$\pm\frac{\sqrt{3}}{3} = \tan x \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$



1

$\frac{d^2s}{dx^2} = -16\sin 4x - 8\sin 2x$

$\frac{d^2s}{dx^2}\left(x = \frac{\pi}{6}\right) = -16\frac{\sqrt{3}}{2} - 8\frac{\sqrt{3}}{2} = -24\frac{\sqrt{3}}{2} = -12\sqrt{3} < 0$

$\cap$ ✓

$\frac{d^2s}{dx^2}\left(x = \frac{5\pi}{6}\right) = -16\left(-\frac{\sqrt{3}}{2}\right) - 8\left(-\frac{\sqrt{3}}{2}\right) = 12\sqrt{3} > 0$

$\cup$

$\therefore$ max of $s(x)$ is for $x = \frac{\pi}{6}$ 1

e) $2 \le a \le 3$

$$\int_0^a \cos(x + a^2)\, dx = \sin a$$

$$\sin(x+a^2)\Big|_0^a = \sin a$$

$$\sin(a+a^2) - \sin a^2 = \sin a \quad (1)$$

using Product to sum trig. id.

$$2\cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$\left.\begin{array}{l} A+B = a+a^2 \\ A-B = a^2 \\ \hline 2A = 2a^2+a \\ A = \frac{2a^2+a}{2} \end{array}\right\} \Rightarrow B = A - a^2 = \frac{2a^2+a-2a^2}{2} = \frac{a}{2}$$

$\therefore$ (1) becomes

$$2\cos\frac{2a^2+a}{2}\sin\frac{a}{2}=\sin a$$

then

using double angle on right

$$2\cos\frac{2a^2+a}{2}\sin\frac{a}{2}=2\sin\frac{a}{2}\cos\frac{a}{2}$$

$$2\sin\frac{a}{2}\left[\cos\frac{2a^2+a}{2}-\cos\frac{a}{2}\right]=0 \quad 1$$

$$2\sin\frac{a}{2}$$

$$2\sin A\sin B=\cos(A-B)-\cos(A+B)$$

$$\left.\begin{aligned}A-B&=\frac{2a^2+a}{2}\\ A+B&=\frac{a}{2}\end{aligned}\right\} B=\frac{a^2+a}{2}-\frac{2a^2+a}{2}=\frac{-a^2}{2}$$

$$2A=\frac{2a^2+2a}{2}=a^2+a$$

$$A=\frac{a^2+a}{2}$$

$$\therefore 2\sin\frac{a}{2}\times 2\sin\frac{a^2+a}{2}\sin\left(\frac{-a^2}{2}\right)=0$$

$$-4\sin\frac{a}{2}\sin\frac{a^2+a}{2}\sin\frac{a^2}{2}=0$$

$\sin\frac{a}{2}\ \sin\frac{a^2+a}{2}\ \sin\frac{a^2}{2}=0$

$\therefore\ \frac{a}{2}=k\pi$ or $\frac{a^2+a}{2}=n\pi$ or $\frac{a^2}{2}=m\pi$

$a=2k\pi$ or $a=\pm\sqrt{2m\pi}$ or $a=\frac{-1\pm\sqrt{1+8n\pi}}{2}$

1

where $k,m,n\in\mathbb{Z}$

as $2\le a\le 3$

all values should be positive

$a^2+a-2n\pi=0$

$a=\frac{-1\pm\sqrt{1-4\times1\times(-2n\pi)}}{2}$

$a=\frac{-1\pm\sqrt{1+8n\pi}}{2}$

now: $2\le a\le 3$ ;

$\therefore\ 2\le 2k\pi\le 3$

$\frac{1}{\pi}\le k\le\frac{3}{2\pi}$

$0.318\le k\le 0.477$

$2\le\frac{-1+\sqrt{1+8n\pi}}{2}\le 3$

$5\le\sqrt{1+8n\pi}\le 7$

$24\le 8n\pi\le 48$

$\frac{24}{8\pi}\le n\le\frac{48}{8\pi}$

$2\le\sqrt{2m\pi}\le 3$

$\frac{4}{2\pi}\le m\le\frac{9}{2\pi}$

$0.64\le m\le 1.432$

$\therefore m=1$

$$\frac{3}{\pi} \leq n \leq \frac{6}{\pi}$$

$$0.95 \leq n \leq 1.91$$

$$\therefore \underline{\underline{n = 1}}$$

Final solution for $a$ is

$$a = \sqrt{2\pi} \text{ or } \frac{-1+\sqrt{1+8\pi}}{2}$$

1

or in set notation

$$a \in \left\{ \sqrt{2\pi}, \frac{-1+\sqrt{1+8\pi}}{2} \right\}$$

# Q12

a) $z_1 = a + (a-3)i$ , $a \in \mathbb{R}^+$

i) $\arg z_1 = \theta$ , $-\frac{\pi}{2} < \theta < 0$

$\arg(-2z_1) = \arg(-2) + \arg(z_1)$

$= \pi + \theta$ ①

b) $\arg(z_1 - 2a) = ?$

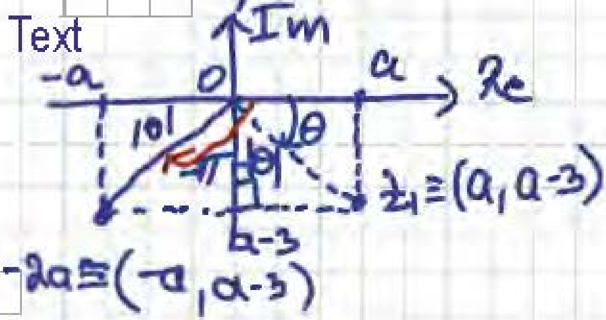
$z_1 - 2a = a + (a-3)i - 2a = -a + (a-3)i$

$\arg(z_1 - 2a) = \arg(-a + (a-3)i)$ ①

$= -\pi + |\theta|$

$= -\pi + (-\theta)$

$= -\pi + \theta$ ①




ii) $z_2 = 1 + 3i$ , find $a$ such that

$$\frac{|z_1 z_2|^2}{\operatorname{Im}(z_1 z_2)} \leq 10$$

---

$z_1 z_2 = (a + (a-3)i)(1 + 3i)$

$= (a - 3a + 9) + (3a + a - 3)i$

$= (-2a + 9) + (4a - 3)i$

$\therefore Im(z_1 z_2) = 4a-3$ 1

$|z_1 z_2|^2 = |z_1|^2 |z_2|^2 = (a^2 + (a-3)^2)(1^2+3^2)$

$= 10(2a^2 - 6a + 9)$ 1

$\therefore \dfrac{|z_1 z_2|^2}{Im(z_1 z_2)} \le 10$

$\dfrac{10(2a^2-6a+9)}{4a-3} \le 10$

$\dfrac{2a^2-6a+9}{4a-3} \le 1$

$\dfrac{2a^2-6a+9 - (4a-3)}{4a-3} \le 0$

$\dfrac{2a^2-6a+9-4a+3}{4a-3} \le 0$

$\dfrac{2a^2-10a+12}{4a-3} \le 0$

$\dfrac{2(a-2)(a-3)}{4a-3} \le 0$ 1


$\therefore a < \frac{3}{4}$ or $2 \le a \le 3$ 1

since $-\frac{\pi}{2} < \arg z_1 < 0 \therefore Im(z_1) = a-3 < 0$, $2 \le a < 3$

also it is given $a > 0 \therefore a < 3 \therefore 0 < a < \frac{3}{4}$ or

b) Let the digit of the number, in order, be $a, b, c, d$

$$N = 1000a + 100b + 10c + d$$ (1)

If $N$ divisible by 11, then $1000a + 100b + 10c + d = 11k$
$k \in \mathbb{Z}$

$1001 \div 11 = 91$

$\therefore 1001a - a + 99b + b + 11c - c + d = 11k$

$11 \times 91a - a + 11 \times 9b + b + 11c - c + d = 11k$

$11 \times 91a + 11 \times 9b + 11c - 11k = a - b + c - d$ (1)

$11(91a + 9b + c - k) = (a+b) - (b+d)$

$\therefore$ if $N$ divisible by 11, the difference of sum of even digits and the sum of odd digits is a multiple of 11. (1)

Q12c) $\cot^3 x$ , $\cot^5 x$

$$\int_{\pi/4}^{\pi/3} \cot^6 x \, dx = ?$$

let $y = \cot^5 x$

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \operatorname{cosec}^2 x$

$y' = -5\cot^4 x \operatorname{cosec}^2 x$

$= -5\cot^4 x \,(1 + \cot^2 x)$

$= -5\cot^4 x - 5\cot^6 x$ ①

$5\cot^6 x = -5\cot^4 x - y'$

$\cot^6 x = -\frac{1}{5} y' - \cot^4 x$

$$\therefore \int_{\pi/4}^{\pi/2} \cot^6 x \, dx = -\left[\frac{1}{5}\int_{\pi/4}^{\pi/2} y' + \int_{\pi/4}^{\pi/2} \cot^4 x \, dx\right]$$

$$= -\frac{1}{5} y \Big|_{\frac{\pi}{4}}^{\pi/2} - \int_{\pi/4}^{\pi/2} \cot^4 x \, dx$$

$$-\frac{1}{5} \cot^5 x \Big|_{\frac{\pi}{4}}^{\pi/2} - \int_{\pi/4}^{\pi/2} \cot^4 x \, dx$$

$= -\frac{1}{5}(0-1) - I$ ①

$I = \int_{\frac{\pi}{4}}^{\pi/2} \cot^4 x\, dx$

$y = \cot^3 x$

$y' = -3\cot^2 x \operatorname{cosec}^2 x = 3\cot^2 x(1+\cot^2 x)$ ①

$= -3\cot^2 x - 3\cot^4 x$

$\therefore \cot^4 x = -\frac{1}{3} y' - \cot^2 x$

$\therefore \int_{\pi/4}^{\pi/2} \cot^4 x\, dx = -\frac{1}{3} y \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\pi/4}^{\frac{\pi}{2}} \cot^2 x\, dx$ ①

$= -\frac{1}{3} \cot^3 x \Big|_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} (\operatorname{cosec}^2 x - 1)\, dx$

$= -\frac{1}{3}(0-1) - (\cot x - x)\Big|_{\pi/4}^{\pi/2}$

$= +\frac{1}{3} - \left[0 - 1 - \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$

$= \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{\pi}{4} + \frac{4}{3} = I$ ①

$\therefore \int_{\pi/4}^{\pi/2} \cot^6 dx = \qquad = \frac{13}{15} - \frac{\pi}{4}$

# Q13

a) i) $k \geq 3$ $(k-1)(k+1) < k^2$

show $\dfrac{1}{(k-1)(k+1)} > \dfrac{1}{k^3}$

Since $(k-1)(k+1) < k^2$, $k \geq 3$

$\therefore (k-1)k(k+1) < k^3$

$\therefore \dfrac{1}{(k-1)k(k+1)} > \dfrac{1}{k^3}$ 1

ii) $S_n = \dfrac{1}{3^3} + \dfrac{1}{4^3} + \dfrac{1}{5^3} + \dots + \dfrac{1}{n^3} = \sum_{3}^{n} \dfrac{1}{k^3}$

to prove $S_n < \dfrac{1}{12}$

Let $\dfrac{1}{(k-1)k(k+1)} = \dfrac{A}{(k-1)k} + \dfrac{B}{k(k+1)} = \dfrac{1/2}{(k-1)k} - \dfrac{\frac{1}{2}}{k(k+1)}$ 1 $(*)$

Now $S_n = \dfrac{1}{3^3} + \dfrac{1}{4^3} + \dfrac{1}{5^3} + \dots + \dfrac{1}{n^3}$

$\therefore S_n < \dfrac{1}{2\times3\times4} + \dfrac{1}{3\times4\times5} + \dfrac{1}{4\times5\times6} + \dots + \dfrac{1}{(n-1)n(n+1)}$ using (i) inequal

$= \sum_{k=3}^{n} \dfrac{1}{(k-1)k(k+1)}$

$\therefore S_n < \dfrac{1}{2} \sum_{k=3}^{n} \left[\dfrac{1}{(k-1)k} - \dfrac{1}{k(k+1)}\right]$ from $(*)$ 1

$\therefore 2S_n < \left[\dfrac{1}{2\times3} - \cancel{\dfrac{1}{3\times4}} + \cancel{\dfrac{1}{3\times4}} - \cancel{\dfrac{1}{4\times5}} + \cancel{\dfrac{1}{4\times5}} \dots\right.$

$$\cdots + \cancel{\frac{1}{(n-1)n}} - \frac{1}{n(n+1)}\Big] = \frac{1}{2\times 3} - \frac{1}{n(n+1)}$$

$$\therefore \quad 2S_n < \frac{1}{6} - \frac{1}{n(n+1)}$$

$$\therefore 2S_n < \frac{1}{6} \quad \text{for } n \geq 3$$

$$\therefore \quad S_n < \frac{1}{12} \qquad 1$$

Q13 b) i) $\frac{1}{p} + \frac{1}{q} = 1$

$\therefore p+q = pq = s$

$(p+q)^2 = p^2+q^2+2pq \quad (1)$

$(p+q)^2 = (pq)^2 = s^2 \quad (2)$ 1

$(1) \,\&\, (2) \Rightarrow (p+q)^2 \geq 4pq \Rightarrow s \geq 4$

$$\frac{1}{p(p+1)} + \frac{1}{q(q+1)} = \frac{1}{p} - \frac{1}{p+1} + \frac{1}{q} - \frac{1}{q+1} =$$

$$= 1 - \frac{p+q+2}{(p+1)(q+1)}$$

$$= 1 - \frac{s+2}{2s+1} = \frac{s-1}{2s+1}$$

$\therefore$ we have to prove

$$\frac{1}{3} \leq \frac{s-1}{2s+1} \leq \frac{1}{2}$$

but $2s+1 \leq 3s-3 \Leftrightarrow 4 \leq s$ and 1

$2s-2 \leq 2s+1 \Leftrightarrow -2 \leq 1 \quad \square$

b) ii)

$$\frac{1}{p(p-1)} + \frac{1}{q(q-1)} = \frac{1}{p-1} - \frac{1}{p} + \frac{1}{q-1} - \frac{1}{q} =$$

$$= \frac{p+q-2}{(p-1)(q-1)} - 1$$ 1

$$= \frac{s-2}{s-s+1} - 1$$

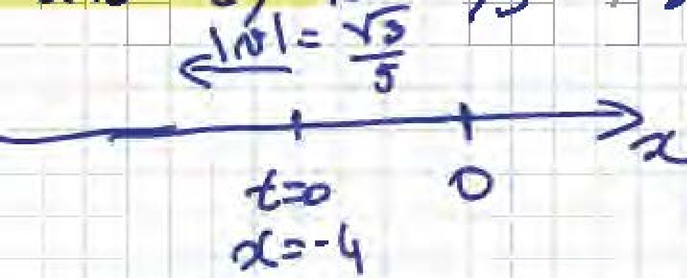
$= s-3$ and as $s \geq 4$ from (i)

$\therefore \; s-3 \geq 1$ 1

$$\therefore \; \frac{1}{p(p-1)} + \frac{1}{q(q-1)} \geq 1$$

Q13 c) $v$ m/s    show $v^2 = 0.01\frac{x^4-64}{x^2}$

$= f(x)$




$t=0, x=-4$ m

$|v| = \frac{\sqrt{3}}{5}$ m/s

$\therefore v = -\frac{\sqrt{3}}{5}$

$\ddot{x} = 0.01\left(x + \frac{64}{x^3}\right) = f(x)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 0.01\left(x + \frac{64}{x^3}\right)$ (1)

$\frac{1}{2}v^2 = 0.01\left(\frac{x^2}{2} - \frac{64}{2x^2}\right) + C$    or

$\frac{1}{2}\int_{-\frac{\sqrt{3}}{5}}^{v} v^2\,dv = 0.01\int_{-4}^{x}\left(x+\frac{64}{x^3}\right)dx$

$v^2 = 0.01\left(x^2 - \frac{64}{x^2}\right) + C$

$x=-4, v = -\frac{\sqrt{3}}{5}$

$\frac{3}{25} = \frac{1}{100}\left(16 - \frac{64}{16}\right) + C$

$12 = 12 + C \Rightarrow C = 0$ (1)

$\therefore v^2 = 0.01\left(\frac{x^4-64}{x^2}\right)$

(ii) why $v = +\frac{\sqrt{x^4-64}}{10x}$

Object will be stationary when $v=0$

$\therefore x^4 - 64 = 0$

$\therefore x = \pm 2\sqrt{2}$

Since object starts at $x=-4$ and moves left it will never come to rest ($x=-2\sqrt{2}$)

$\therefore$ velocity is always negative, so when we take the $\sqrt{\ }$ we need to choose an expression that is always negative and since $x<0$ $\therefore v = \dfrac{\sqrt{x^4-64}}{10x}$

or For $x<0$, $x$ and $\frac{1}{x^3}$ are both negative

$\therefore$ the acceleration is always negative for $x<0$ and since the initial direction of motion is to the left from position $x=-4$ then the object will always move towards the left, $\therefore$ the velocity will be always negative. Therefore choose the positive

$\sqrt{x^2-64}$ & $10x<0$

iii) $t = ?$ $x = -50m$

$$v = \frac{\sqrt{x^4 - 64}}{10x}$$

$$\frac{dx}{dt} = \frac{\sqrt{x^4 - 64}}{10x}$$ 1

$$\frac{dt}{dx} = \frac{10x}{\sqrt{x^4 - 64}}$$

$$\int_0^t dt = \int_{-4}^{-50} \frac{10x}{\sqrt{x^4 - 64}} dx$$

let $u = x^2$ $x = -4$, $u = 16$ 1

$x = 50$, $u = 2500$

$du = 2x\,dx$

$$t = \int_{16}^{2500} \frac{5du}{\sqrt{u^2 - 64}}$$

RCR for $\int \frac{f'(x)}{\sqrt{f(x)^2 - a^2}} = \ln\left|f(x) + \sqrt{f(x)^2 - a^2}\right| + C$

$$= 5\left[\ln\left|u + \sqrt{u^2 - 64}\right|\right]_{16}^{2500}$$ 1

$$= 5\left[\ln(2500 + \sqrt{2500^2 - 64^2}) - \ln(16 + \sqrt{16^2 - 64})\right]$$

$\approx 25.604$ sec

time, correct to nearest second, to reach a position 50m to the left of the origin is 26 seconds. 1

13(a)

(i) $(k-1)(k+1) < k^2$

Multiplying both sides by $k$ ($k \geq 3$ given) } or equivalent working

$(k-1)k(k+1) < k^3$

$\Rightarrow \frac{1}{(k-1)k(k+1)} > \frac{1}{k^3}$

(ii) Let $f(k) = -\frac{1}{2k(k+1)}$

$$\Rightarrow f(k) - f(k-1) = \frac{1}{2(k-1)(k)} - \frac{1}{2k(k+1)}$$

$$= \frac{(k+1)-(k-1)}{2(k-1)k(k+1)}$$

$$= \frac{1}{(k-1)k(k+1)}$$

$\therefore \frac{1}{k^3} < \frac{1}{k(k-1)(k+1)} \Rightarrow \frac{1}{k^3} < f(k) - f(k-1)$

$$\Rightarrow \frac{1}{3^3} < f(3) - f(2)$$

$$\frac{1}{4^3} < f(4) - f(3)$$

$$\frac{1}{5^3} < f(5) - f(4)$$

$$\vdots$$

$$\frac{1}{(n-2)^3} < f(n-2) - f(n-3)$$

$$\frac{1}{(n-1)^3} < f(n-1) - f(n-2)$$

$$\frac{1}{n^3} < f(n) - f(n-1)$$

falling sum

$\Rightarrow$ Adding all $S_n < f(n) - f(2)$

$$= -\frac{1}{2n(n+1)} - \frac{1}{2 \cdot 2 \cdot (3)}$$

$$= \frac{1}{12} - \frac{1}{2n(n+1)}$$

$$< \frac{1}{12}$$

13(b) $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow \boxed{p+q=pq}$ and since $\frac{1}{p}+\frac{1}{q} \geq \frac{2}{\sqrt{pq}}$ (AM≥GM)

$\Rightarrow 1 \geq \frac{2}{\sqrt{pq}}$

$\Rightarrow \boxed{pq \geq 4}$

let $t = \frac{1}{p(p+1)} + \frac{1}{q(q+1)}$

$= \frac{1}{p} - \frac{1}{p+1} + \frac{1}{q} - \frac{1}{q+1}$

$= 1 - \left(\frac{1}{p+1} + \frac{1}{q+1}\right) \quad (\because \frac{1}{p}+\frac{1}{q}=1)$

$= 1 - \frac{2+pq}{1+2pq} \quad (\because p+q=pq)$

$= \frac{1}{2} - \frac{3}{2(1+2pq)}$

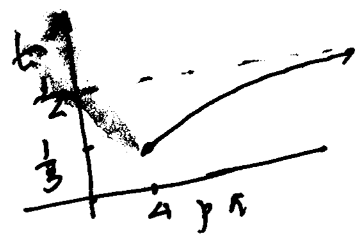
$t$ is continuous $\forall\, pq$

$\frac{dt}{d(pq)} = \frac{3}{(1+2pq)^2}$

$\Rightarrow t$ is monotonic increasing w.r.t $pq$

when $pq=4$ $\quad t_1 = \frac{1}{2} - \frac{3}{2(1+8)} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

as $pq \to \infty$ $\quad t \to \frac{1}{2}$




$\Rightarrow \frac{1}{3} \leq t < \frac{1}{2}$

ie, $\frac{1}{3} \leq \frac{1}{p(p+1)} + \frac{1}{q(q+1)} < \frac{1}{2}$

13(b)(ii) $\frac{1}{p(p-1)} + \frac{1}{q(q-1)}$

$= \frac{1}{p-1} - \frac{1}{p} + \frac{1}{q-1} - \frac{1}{q}$

$= \frac{1}{p-1} + \frac{1}{q-1} - 1$ $\left(\because \frac{1}{p} + \frac{1}{q} = 1\right)$

$= \frac{pq-2}{pq-p-q+1} - 1$

$= \frac{pq-2}{1} - 1$

$= pq - 3$

$\geq 1$ (since $pq \geq 4$ from (i))

13(c) (ii) $\ddot{x} = 0.01\left(x + \frac{64}{x^3}\right)$

$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 0.01\left(x + \frac{64}{x^3}\right)$

$\int d\left(\frac{1}{2}v^2\right) = \int 0.01\left(x + \frac{64}{x^3}\right)dx$

$\frac{v^2}{2} = 0.01\left(\frac{x^2}{2} - \frac{32}{x^2}\right) + C$

Initially $x = -4$, $v = -\frac{\sqrt{3}}{5}$

$\Rightarrow C = \frac{3}{50} - 0.01(8-2) = 0$

$\Rightarrow v^2 = \frac{0.01}{x^2}(x^4 - 64)$

$v = \pm \cancel{\phantom{xx}} \pm \frac{\sqrt{x^4-64}}{10x}$

$\therefore$ Initially $v < 0$, $x < 0 \Rightarrow v = \frac{\sqrt{x^4-64}}{10x}$

$v = -\frac{\sqrt{x^4-64}}{10x}$ is not acceptable as it contradicts the initial condition

13 (c) (iii)

$$v = \frac{\sqrt{x^4-64}}{10x}$$

$$\frac{dx}{dt} = \frac{\sqrt{x^4-64}}{10x}$$

$$\Rightarrow \int_{-4}^{-50} \frac{10x}{\sqrt{x^4-64}}\,dx = \int_0^t dt$$

$$\Rightarrow \left[5\ln\left[x^2+\sqrt{x^4-64}\right]\right]_{-4}^{-50} = t$$

$$\Rightarrow t = 26.220\ldots$$

$$\approx \underline{26 \text{ min}}$$

Q14 a) $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$ &

$a_1 \times a_2 \times a_3 \times \cdots \times a_n = 1$

Prove $(a_1^2+a_1)(a_2^2+a_2) \ldots (a_n^2+a_n) \geq 2^n$

---

$a_1 \geq \sqrt{a_1}$

$(a_1 - \sqrt{a_1})^2 \geq 0$

$a_1^2 - 2a_1\sqrt{a_1} + a_1 \geq 0$

$a_1^2 + a_1 \geq 2a_1\sqrt{a_1}$ (1)

Similarly for $a_2$ to $a_n$ 1

$a_2^2 + a_2 \geq 2a_2\sqrt{a_2}$ (2)

$\vdots$

$a_n^2 + a_n \geq 2a_n\sqrt{a_n}$ (n) 1

(1) × (2) × … × (n) :

$\therefore (a_1^2+a_1)(a_2^2+a_2)(a_3^2+a_3) \cdots (a_n^2+a_n) \geq$

$2 \times a_1\sqrt{a_1} \times 2a_2\sqrt{a_2} \times \cdots \times 2a_n\sqrt{a_n}$

$\geq 2^n(a_1 a_2 \ldots a_n)\sqrt{a_1 a_2 \times \cdots a_n}$

1

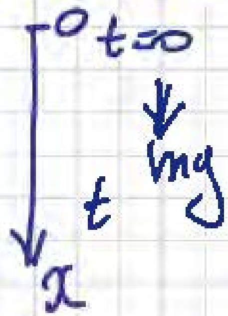
$\geq 2^n$

Q14 b)

i) $m = 90\,kg$ $\quad h = 100\,m$ $\quad g = 10\,m/s^2$

$v = f(x) = ?$ $\quad \boxed{v_0 = u}$

α) no resistance




$$m\ddot{x} = mg$$

$$\ddot{x} = g$$

$$v\frac{dv}{dx} = g$$

$$\int_u^v v\,dv = \int_0^x 10\,dx \qquad 1$$

$$\frac{v^2}{2}\Big|_u^v = 10x\Big|_0^x$$

$$\frac{v^2 - u^2}{2} = 10x$$

$$\frac{v^2}{2} = 10x + \frac{u^2}{2}$$

$$v^2 = 20x + u^2 \Rightarrow v = \sqrt{u^2 + 20x}\ ,\ v > 0 \qquad 1$$

(a) $m\ddot{x} = mg - 0.27v^2$

$$\ddot{x} = g - \frac{0.27}{90}v^2$$

$$\ddot{x} = 10 - 0.003v^2 \quad 1 \quad (1)$$

$$v\frac{dv}{dx} = 10 - 0.003v^2$$

$$\int_0^x dx = \int_u^v \frac{v\,dv}{10 - 0.003v^2}$$

1

$$x = -\frac{1}{0.003}\,\frac{1}{2}\Big[\ln|10 - 0.003v^2|\Big]_u^v$$

$$= -\frac{1}{0.006}\ln\frac{10 - 0.003v^2}{10 - 0.003u^2}$$

$$\ln\frac{10 - 0.003v^2}{10 - 0.003u^2} = -0.006x$$

$$\frac{10 - 0.003v^2}{10 - 0.003u^2} = e^{-0.006x}$$

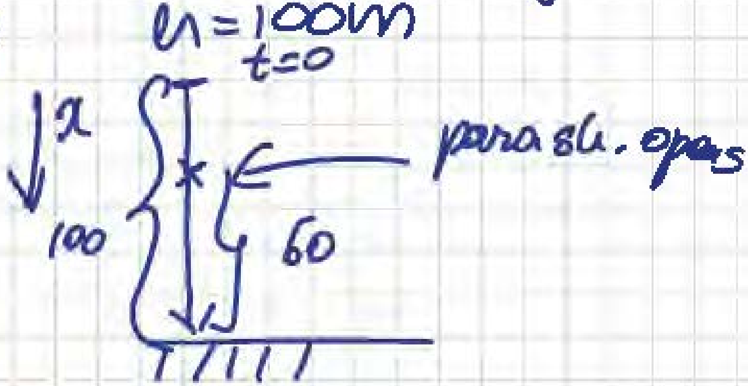
$$(10 - 0.003v^2) = (10 - 0.003u^2)\,e^{-0.006x}$$

1

$$v = \sqrt{\frac{10 - (10 - 0.003u^2)\,e^{-0.006x}}{0.003}}, \ v > 0$$

5) (i) $t=0$, $v=0$, 60 m from the ground

(a) $v = \dot{x} = ?$ after 40m




$x = 40\text{ m}$

from i)(a):

$$v = \sqrt{u^2 + 20x}$$

$$v = \sqrt{0^2 + 20 \times 40}$$

$$= 20\sqrt{2} \text{ m/s}$$

$$\approx 28.28 \text{ m/s}$$

1

(b) $v = \dot{x} = ?$ on landing, $x = 60\text{ m}$, $u = 20\sqrt{2}$

from i)(b) (now with resistance) 1

$$v = \sqrt{\frac{10 - (10 - 0.003 \times (20\sqrt{2})^2)\, e^{-0.006 \times 60}}{0.003}}$$

$$v = \sqrt{\frac{10 - (10 - 0.003 \times 800)\, e^{-0.36}}{0.003}}$$

$$v = \sqrt{\frac{10 - 5.3023}{0.003}} = \sqrt{1565.8866}$$

$$v = 39.571285 \approx 39.6 \text{ m/s}$$ 1

5) (c) $v_L = 39.6$ (from 6)(b))

$\frac{v_L}{V_T} = ?$

$v_T = ?$ when $\ddot{x} = 0$

from (2)(i)(b) $\ddot{x} = 10 - 0.003v^2 \Rightarrow$ (1)

$$v_T = \sqrt{\frac{10}{0.003}}$$

$$v_T = 57.73502 \ \frac{m}{s}$$
$$= 57.7 \ m/s$$

$$\frac{v_L}{V_T} = \frac{39.6}{57.7} \quad 100\% = 68.539\%$$

$$\approx 69\%$$ (1)

14 c)

$a, b, c \in \mathbb{Z}^+$

$a^2 + b + c$, $b^2 + a + c$, $c^2 + a + b$ can't be perfect squares

Let assume $a \le b \le c$

$\therefore c^2 < c^2 + a + b \le c^2 + 2c < (c+1)^2$ 1

$\therefore c^2 + a + b < (c+1)^2$

$\therefore c^2 + a + b$ cannot be perfect square

Similarly for the other two. 1

Q15

a) $r = (2+\lambda)\underset{\sim}{i} + (3-\lambda)\underset{\sim}{j} + (4-2\lambda)k$

i) in $xy$ plane: $z=0$, $4-2\lambda=0 \Rightarrow \lambda=2$

point $A(4, 1, 0)$ ①

in $xz$ plane: $y=0$, $3-\lambda=0 \Rightarrow \lambda=3$

point $B(5, 0, -2)$ ①

in $yz$ plane: $x=0$, $2+\lambda=0 \Rightarrow \lambda=-2$

point $C(0, 5, 8)$ ①

iii) $\text{proj}_{\vec{AB}}\,\vec{OC} = \dfrac{\vec{OC}\cdot\vec{AB}}{|\vec{AB}|^2}\,\vec{AB}$

$\vec{AB} = \begin{pmatrix}1\\-1\\-2\end{pmatrix}$

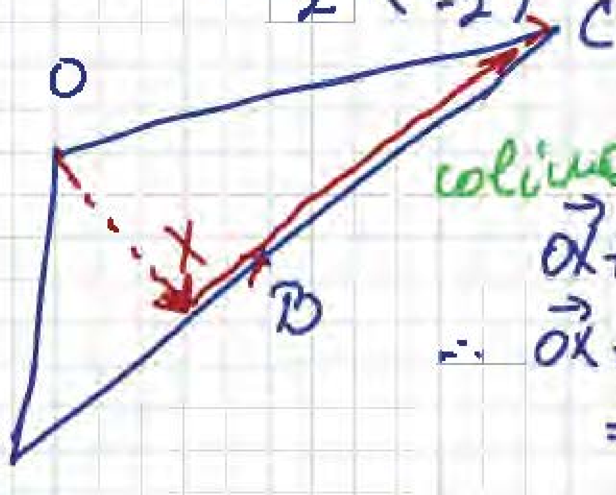
$\vec{OC} = \begin{pmatrix}0\\5\\8\end{pmatrix}$

$= \dfrac{(0-5-16)}{\left(\sqrt{1+1+4}\right)^2}\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

$= \dfrac{-21}{6}\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ (1)

$= -\dfrac{7}{2}\begin{pmatrix}1\\-1\\-2\end{pmatrix} = \vec{XC}$

iv)




if not used colinearity of A, B, C no mark given

$\vec{OX} + \vec{XC} = \vec{OC}$

$\therefore \vec{OX} = \vec{OC} - \vec{XC}$

$= \begin{pmatrix}0\\5\\8\end{pmatrix} + \dfrac{7}{2}\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

Area $\triangle OAC = \dfrac{1}{2}|\vec{AC}| \times |\vec{OX}|$ (1)

$\vec{AC} = \begin{pmatrix}-4\\4\\8\end{pmatrix}$ $\quad \vec{OX} = \begin{pmatrix}7/2\\5-\frac{7}{2}\\8-7\end{pmatrix} = \begin{pmatrix}7/2\\\frac{3}{2}\\1\end{pmatrix} = \dfrac{1}{2}\begin{pmatrix}7\\3\\2\end{pmatrix}$

$$\text{Area } \Delta_{OAC} = \frac{1}{2}\sqrt{16+16+64} \times \sqrt{\frac{1}{4}(49+9+4)}$$

$$= \frac{1}{2}\sqrt{98}\ \sqrt{\frac{1}{4}\ 62}$$

$$= \frac{1}{4}\ 4\sqrt{6}\ \sqrt{62}$$

$$= \sqrt{3\times 2\times 2\times 31}$$

point to use part iii)

$$= 2\sqrt{93}$$

$$= 19.2873052$$

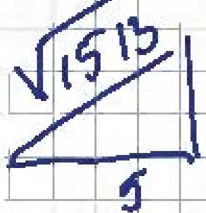
$$\boxed{\approx 19.3\ (3\text{ s.f.})}$$ 1

or $A = \frac{1}{2}|\vec{OA}||\vec{OC}|\sin(\angle COA)$

$$\vec{OC}\cdot\vec{OA} = 5 = |\vec{OA}||\vec{OC}|\cos(\angle COA)$$

$$= \sqrt{17}\sqrt{89}\cos(\angle COA)$$

$$\cos(\angle COA) = \frac{5}{\sqrt{1513}} \Rightarrow \sin(\angle COA) = \frac{\sqrt{1488}}{\sqrt{1513}}$$




$$\therefore A = \frac{1}{2}\sqrt{1513}\ \frac{\sqrt{1488}}{\sqrt{1513}} = 2\sqrt{93} \approx 19.3$$

15b) $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n \, dx \quad , n \geq 0$

i) $\underline{n \geq 2 \quad , I_n = \left(\frac{n-1}{n}\right) I_{n-2}}$

use only trig. fun. when int. by parts

$I_n = \int_0^{\pi/2} (\sin x)^{n-1} \frac{d}{dx}(-\cos x)\, dx$

$= \left[(\sin x)^{n-1} \cdot (-\cos x)\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} (-\cos x) \frac{d}{dx}(\sin x)^{n-1} dx$ ①

$= [0 - 0] + (n-1) \int_0^{\pi/2} (\sin x)^{n-2} \cos x \cos x \, dx$

$= (n-1) \int_0^{\pi/2} (\sin x)^{n-2} \cos^2 x \, dx$

$= (n-1) \int_0^{\pi/2} (\sin x)^{n-2} (1 - \sin^2 x) \, dx$ ①

$= (n-1) \int_0^{\pi/2} (\sin x)^{n-2} dx - (n-1) \int_0^{\pi/2} \sin^n x \, dx$

$\therefore I_n = (n-1) I_{n-2} - (n-1) I_n$

$\therefore I_n (1 + (n-1)) = (n-1) I_{n-2}$ ①

$\therefore \boxed{I_n} = \frac{n-1}{n} I_{n-2} \quad , n \geq 2 \; \square$

ii) from i) $I_{2n} = \left(\frac{2n-1}{2n}\right) I_{2n-2}$

$$= \frac{2n-1}{2n}\left(\frac{2n-3}{2n-2} I_{2n-4}\right) \text{ re-using the i) results}$$

$$= \frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\,\frac{2n-5}{2n-4}\, I_{2n-6}$$

$$= \frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\,\frac{2n-5}{2n-4}\cdots\frac{(2n-2n)+1}{(2n-2n)+1}\, I_{2n-2n}$$

$$= \frac{(2n-1)(2n-3)(2n-5)\cdots 1}{(2n)(2n-2)(2n-4)\cdots 2}\, I_0 \quad ①$$

$$I_0 = \int_0^{\pi/2} (\sin x)^0\, dx = x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore I_{2n} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad ①$$

Also from (i) $I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$

$$= \frac{2n}{2n+1}\left(\frac{2n-2}{2n-1} I_{2n-3}\right) \text{ using (i) again}$$

$$= \frac{2n}{2n+1}\,\frac{2n-2}{2n-1}\,\frac{2n-4}{2n-3}\, I_{2n-5} \text{ again (i)}$$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdot\dots\cdot\frac{2}{3}\cdot I_1$$

$$I_1 = \int_0^{\pi/2} \sin x\,dx = -\cos x\Big|_0^{\pi/2}$$

$$= -(0-1) = 1$$

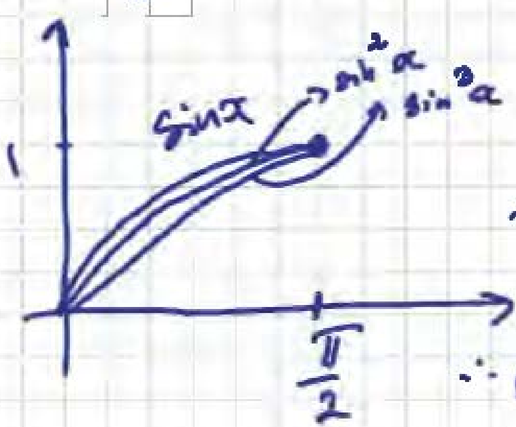
1

$$\therefore I_{2n+1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdot\dots\frac{4}{5}\cdot\frac{2}{3}\cdot 1$$

iii) $I_k = \int_0^{\pi/2} (\sin x)^k\,dx$

$I_{k+1} = \int_0^{\pi/2} (\sin x)^{k+1}\,dx$ $\Big\}$ $k \geq 0$

$I_k$ & $I_{k+1}$ represent the area under the curves $y_k = (\sin x)^k$ & $y_{k+1} = (\sin x)^{k+1}$ for $0 \leq x \leq \frac{\pi}{2}$




for $0 \leq x \leq \frac{\pi}{2}$

$0 \leq \sin x \leq 1$

$\therefore (\sin x)^{k+1} \leq (\sin x)^k$

$\therefore$ area under the

curve $y_k$ is greater than the area under the curve $y_{k+1}$ for $0 \le x \le \frac{\pi}{2}$

$\therefore I_k > I_{k+1}$ (1)

iv) using the results in (ii) for $I_{2n+1}$

$$I_{2n-1} = \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \frac{2n-6}{2n-5} \cdots \frac{4}{5} \frac{2}{3} \cdot 1$$

from (iii), $I_k > I_{k+1}$ and $I_{2n-1} > I_{2n}$ and $I_{2n-1} > I_{2n+1}$

since $I_{2n-1} > I_{2n}$, then

$$\frac{2n-2}{2n-1} \frac{2n-4}{2n-3} \cdots \frac{4}{5} \frac{2}{3} \cdot \frac{1}{1} > \frac{2n-1}{2n} \frac{2n-3}{2n-2} \cdots \frac{3}{4} \frac{1}{2} \cdot \frac{\pi}{2}$$

Thus:

$$\frac{(2n)(2n-2)^2(2n-4)^2 \cdots 4^2 \cdot 2^2}{(2n-1)^2(2n-3)^2 \cdots 5^2 3^2 1} > \frac{\pi}{2} \quad (1)$$

hence

$$\frac{2n}{2n+1} \frac{2n(2n-2)^2(2n-4)^2 \cdots 4^2 2^2}{(2n-1)^2(2n-3)^2 \cdots 5^2 3^2 \cdot 1} > \frac{2n}{2n+1} \frac{\pi}{2}$$

i.e. $$\frac{2^2 \cdot 4^2 \cdot \cdots \cdot (2n)^2}{1 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2(2n+1)} > \frac{2n}{2n+1} \frac{\pi}{2} \quad (1)$$

Since $I_{2n} > I_{2n+1}$, then

$$\underbrace{\frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\,\frac{2n-5}{2n-4}\cdots\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}}_{(*)} > \frac{2n}{2n+1}\,\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{4}{5}\,\frac{2}{3}\cdot 1$$

$$(*)\ \frac{\pi}{2} > \frac{(2n)^2(2n-2)^2\cdots 4^2\cdot 2^2}{(2n+1)(2n-1)^2(2n-3)^2\cdots 3^2\cdot 1}$$

$$(*)\ \frac{\pi}{2} > \frac{2^2\cdot 4^2\cdots(2n)^2}{1\cdot 3^2\cdot 5^2\cdots(2n-1)^2(2n+1)} \quad (2)$$

Now from (1) and (2)

1

$$\therefore\ \frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2\cdot 4^2\cdots(2n)^2}{1\cdot 3^2\,5^2\cdots(2n-1)^2(2n+1)} < \frac{\pi}{2}$$

$\square$

or to look at

$$I_{2n+1}\cdot I_{2n} < I_{2n+1}\,I_{2n-1} < I_{2n}\,I_{2n-1}$$

Q16 a)

$z = \frac{1}{4}(1+\sqrt{3}i)$

i) $|1+\sqrt{3}| = \sqrt{1+3} = \sqrt{4} = 2$ (1)

$\arg(1+\sqrt{3}i) = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$

$\therefore \quad z = \frac{1}{4}\left(2e^{i\pi/3}\right) = \frac{1}{2}e^{i\pi/3}$ (1)

ii) $z^2 = \left(\frac{1}{2}e^{i\pi/3}\right)^2 = \frac{1}{4}e^{i\frac{2\pi}{3}}$ (1)

$z^3 = \left(\frac{1}{2}e^{i\pi/3}\right)^3 = \frac{1}{8}e^{i\pi}$ (1)

iii) $A_r \triangle OP_0P_1 = \frac{1}{2}\times(1)\times\left(\frac{1}{2}\right)\sin\frac{\pi}{3}$ (1)

$= \frac{\sqrt{3}}{8}$ units$^2$

$A_r OP_1P_2 = \frac{1}{2}\times\left(\frac{1}{2}\right)\times\left(\frac{1}{4}\right)\sin\frac{\pi}{3} = \frac{\sqrt{3}}{32}$ units$^2$ (1)

$A_r \triangle OP_2P_3 = \frac{1}{2}\times\left(\frac{1}{4}\right)\times\left(\frac{1}{8}\right)\times\sin\frac{\pi}{3} = \frac{\sqrt{3}}{128}$ units$^2$ (1)

$\therefore$ Area of (3+1)-polygon

$= \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{32} + \frac{\sqrt{3}}{128} = \frac{21\sqrt{3}}{128}$ units$^2$

(1)

iv) $\frac{1}{2}(1)\left(\frac{1}{2}\right)\sin\frac{\pi}{n} + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\sin\frac{\pi}{n} +$

$+ \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)\sin\frac{\pi}{n} \cdots + \frac{1}{2}\left(\frac{1}{2^{n-1}}\right)\left(\frac{1}{2^{n}}\right)\sin\frac{\pi}{n}$ 1

$= \frac{1}{2}\left(1\times\frac{1}{2} + \frac{1}{2}\frac{1}{4} + \frac{1}{4}\frac{1}{8} + \cdots + \frac{1}{2^{n-1}}\frac{1}{2^{n}}\right) \times \sin\frac{\pi}{n}$

$= \frac{1}{2}\Big(\underbrace{\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2n-1}}}_{n \text{ terms}}\Big)\sin\frac{\pi}{n}$ 1

Area of $(n+1)$-polygon:

$S_n = a\,\frac{1-r^n}{1-r}$

$\therefore = \frac{1}{2}\left(\frac{\frac{1}{2}\left(1-\left(\frac{1}{2^2}\right)^n\right)}{1-\frac{1}{2^2}}\right)\sin\frac{\pi}{n}$

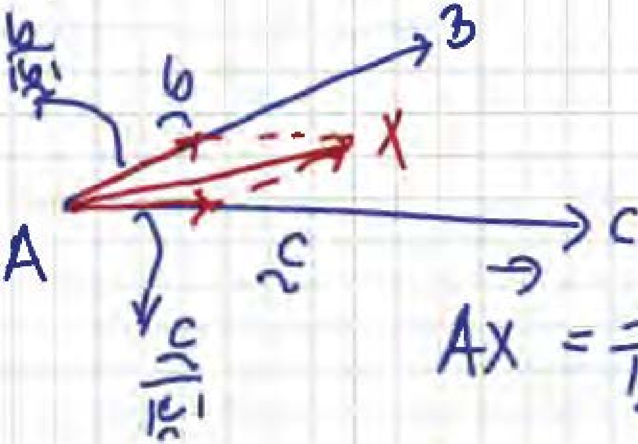
$= \frac{1}{4}\,\frac{4}{3}\left(1-\left(\frac{1}{4}\right)^n\right)\sin\frac{\pi}{n}$

$= \frac{1}{3}\left(1-\left(\frac{1}{4}\right)^n\right)\sin\frac{\pi}{n}$ 1

$\therefore a = \frac{1}{3},\ b = \frac{1}{4}$

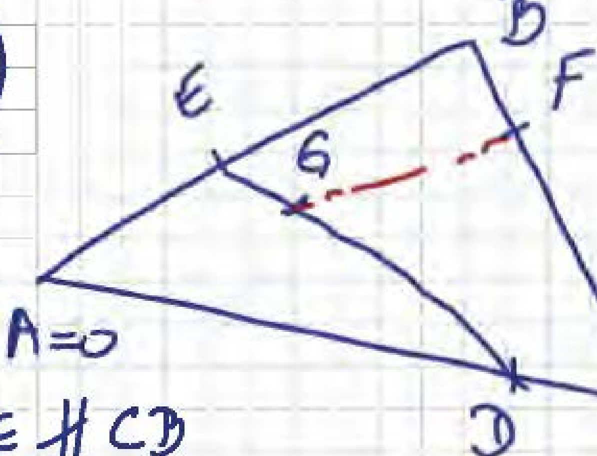
(G5)

i)




$$\vec{AX} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} + \frac{\underset{\sim}{c}}{|\underset{\sim}{c}|}$$ 1

ii)




Set origin at A:

$\vec{AB} = \vec{OB} = \underset{\sim}{b}$

$\vec{AC} = \vec{OC} = \underset{\sim}{c}$

$\vec{AD} = \vec{OD} = \underset{\sim}{d}$

$\vec{AE} = \vec{OE} = \underset{\sim}{e}$

$\vec{AG} = \vec{OG} = \underset{\sim}{g}$

$\vec{AF} = \vec{OF} = \underset{\sim}{f}$

DE // CB

BF : FC = EG : GD = BE : CD

Show GF // AX

---

$$\left.\begin{array}{ll} \underset{\sim}{e} = p\,\underset{\sim}{b} & (1) \\ \underset{\sim}{d} = q\,\underset{\sim}{c} & (2) \end{array}\right\} \quad p, q \in (0, 1) \quad \text{let } \frac{\vec{BF}}{\vec{FC}} = t$$

$$\left.\begin{aligned}\vec{BF} + \underset{\sim}{b} &= \underset{\sim}{f} \Rightarrow t\vec{FC} + \underset{\sim}{b} = \underset{\sim}{f}\\ \vec{FC} + \underset{\sim}{f} &= \underset{\sim}{c}\end{aligned}\right\} \Rightarrow t(\underset{\sim}{c} - \underset{\sim}{f}) + \underset{\sim}{b} = \underset{\sim}{f}$$

$$\therefore\ t\underset{\sim}{c} - t\underset{\sim}{f} + \underset{\sim}{b} = \underset{\sim}{f} \Rightarrow \underset{\sim}{f} = \frac{t\underset{\sim}{c} + \underset{\sim}{b}}{1+t} \quad (3)$$

1

also $\frac{\vec{EG}}{\vec{GD}} = t \Rightarrow \vec{EG} = t\vec{GD}$

$$\left.\begin{aligned}\underset{\sim}{e} + \vec{EG} &= \underset{\sim}{g} \Rightarrow \underset{\sim}{e} + t\vec{GD} = \underset{\sim}{g}\\ \underset{\sim}{g} + \vec{GD} &= \underset{\sim}{d}\end{aligned}\right\} \Rightarrow \underset{\sim}{e} + t(\underset{\sim}{d} - \underset{\sim}{g}) = \underset{\sim}{g}$$

$\therefore\ \underset{\sim}{e} + t\underset{\sim}{d} - t\underset{\sim}{g} = \underset{\sim}{g} \Rightarrow \underset{\sim}{g} = \frac{\underset{\sim}{e} + t\underset{\sim}{d}}{1+t}$ then from

(1) & (2) $\underset{\sim}{g} = \frac{p\underset{\sim}{b} + tq\underset{\sim}{c}}{1+t} \quad (4)$

1

also $\frac{\vec{BE}}{\vec{CD}} = t \Rightarrow |\vec{BE}| = t|\vec{CD}|$

$$|\vec{BE}| = |\underset{\sim}{b} - \underset{\sim}{e}| = |\underset{\sim}{b} - p\underset{\sim}{b}| = (1-p)|\underset{\sim}{b}|$$

$$|\vec{CJ}| = |\underset{\sim}{c} - \underset{\sim}{q}| = |\underset{\sim}{c} - g\underset{\sim}{c}| = (1-g)|\underset{\sim}{c}|$$

$$\therefore (1-p)|\underset{\sim}{b}| = t(1-g)|\underset{\sim}{c}| \quad (5)$$

then using (3) and (4)

$$\vec{GF} = \underset{\sim}{f} - \underset{\sim}{g} = \frac{t\underset{\sim}{c} + b}{1+t} - \frac{p\underset{\sim}{b} + tg\underset{\sim}{c}}{1+t}$$

$$\underset{\sim}{f} - \underset{\sim}{g} = \frac{t - tg}{1+t}\underset{\sim}{c} + \frac{1-p}{1+t}\underset{\sim}{b}$$

$$= \frac{t(1-g)}{1+t}\underset{\sim}{c} + \frac{1-p}{1+t}\underset{\sim}{b}$$

$$\text{and } (5) \Rightarrow t(1-g) = \frac{(1-p)|\underset{\sim}{b}|}{|\underset{\sim}{c}|}$$

$$\therefore \underset{\sim}{f} - \underset{\sim}{g} = \frac{(1-p)}{1+t}\frac{|\underset{\sim}{b}|}{|\underset{\sim}{c}|}\underset{\sim}{c} + \frac{1-p}{1+t}\underset{\sim}{b}$$

$\therefore$

$$\vec{GF} = \frac{1-p}{1+t}|\underset{\sim}{b}|\left(\frac{\underset{\sim}{c}}{|\underset{\sim}{c}|} + \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}\right)$$

1

$$\therefore \vec{GF} = \text{const.} \cdot \vec{AX} \quad \therefore \vec{GF} \parallel \vec{AX} \quad \square$$